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Unitary Propagators for Time-Dependent Hamiltonians with Singular Potentials

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We prove that a unitary propagator $U(t, s)$ for the time-dependent Schrödinger equation $du/dt = iH(t)u$ exists in $L^2(\mathbb{R})$, where $H(t) = -\Delta + \sum_{i=1}^{N} c_i V(x - v_i t)$, $c_i, v_i \in \mathbb{R}$, $v_i \neq v_j$ for $i \neq j$, and V is a distribution with bounded Fourier transform. This extends earlier work with I. Segal on the time-independent case. Such Hamiltonians include, for example, the case of finitely many moving delta potentials. We also apply a method of Segal to study singular time-dependent perturbations of $(1/i)(d/dx)$ in one space dimension and their corresponding unitary propagators, and show that the propagator depends continuously on the potential in a suitable sense. © 1995 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

In earlier work with I. Segal (cf. [4]), self-adjointness of singular perturbations of the form $H_0 + V$, where H_0 is a differential operator acting in $L^2(\mathbb{R})$ and V is a suitable distribution potential, was established by solving the time-dependent Schrödinger equation in the interaction representation. Here we extend those results to the case of *time-dependent* potentials $V(t)$, by proving the existence of a unitary propagator for the time-dependent Schrödinger equation, with Hamiltonian $H_0 + V(t)$. Due to the singular nature of the potentials, which correspond to quadratic form-perturbations of the Laplacian, standard methods such as time-ordered exponential series (i.e., the Dyson expansion), or product integrals, are not directly applicable. More explicitly, we study time-dependent Hamiltonians of the form

$$-\Delta + \sum_{i=1}^{N} c_i V(x - v_i t),$$

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where c_i, $v_i \in \mathbb{R}$, and $v_i \neq v_j$ for $i \neq j$, and V is a distribution with bounded Fourier transform, thereby extending the question of existence of a unitary propagator to finitely many time-dependent singular potentials. As a special case, we obtain results for two moving delta potentials (cf. [2, 7]). Using a modification of the method developed in [4] for time-independent potentials, we first study the time-dependent Schrödinger equation in the interaction representation. We prove in Theorem 1 that the time-ordered operators are bounded in $L^2(\mathbb{R})$, and in Theorem 8 and its corollary, that a unitary propagator $U(t, s)$ exists, and satisfies the identity $U(t, r)U(r, s) = U(t, s)$ for all $r, t, s \in \mathbb{R}$. That is, at least in a formal sense (and in the classical sense when V is sufficiently regular (cf. [8])), $U(t, s)$ satisfies the time-dependent Schrödinger equation

$$\frac{du}{dt} = iH(t)u(t),$$

where $H(t) = -\Delta + \sum_{i=1}^{N} c_i V(x - v_i t)$. The proof is technical and involves an inductive argument involving the kernels of the terms of the time-ordered series. In Section 3, time-dependent singular perturbations of $(1/i)(d/dx)$ are considered and, again using a method of Segal [9], we show that a unitary propagator exists, and depends continuously on the potential V. This permits us to consider Hamiltonians (formally) of the form

$$\frac{1}{i}\frac{d}{dx} + \sum_{i=1}^{N} c_i V(x - v_i t),$$

where, for example, V is a delta potential, as well as to show that the corresponding propagators depend continuously on V, in the sense of distributions.

We begin by reviewing the basic features of the interaction representation approach in [4], and then consider the case $N = 2$ in detail, since the difficulties inherent in this case are typical of the general case.

Let $H_0 = -\Delta$ on $L^2(\mathbb{R})$, and $\mathbb{V}(t) = c_1 V(x - V(x - v_1 t) + c_2 V(x - v_2 t)$, where $c_1, c_2, v_1, v_2 \in \mathbb{R}$, $v_1 \neq v_2$, and V is a distribution satisfying $\hat{V} \in L^\infty(\mathbb{R})$, and $\hat{V}(-k) = \hat{V}(k)$. Consider the interaction Hamiltonian $H_I(t) = e^{-itH_0}\mathbb{V}(t)e^{itH_0}$. Using the Dyson expansion, we shall prove existence of a unitary propagator $U_I(t, s)$ for $H_I(t)$, from which we obtain the propagator $U(t, s) = e^{itH_0}U_I(t, s)$ for the time-dependent Hamiltonian $H_0 + \mathbb{V}(t)$; $U(t, s)$ is a propagator in the usual sense (cf. [8]), when V is sufficiently regular. This resembles the method used in [4] for potentials that do not depend on time.

Taking Fourier transforms, we see that the terms in the Dyson expansion are integral operators with kernels

$$K_{1,t}(x,y) = \left[W_t\left(\left(x+\frac{v_1}{2}\right)^2 - \left(y+\frac{v_1}{2}\right)^2\right) + W_t\left(\left(x+\frac{v_2}{2}\right)^2 - \left(y+\frac{v_2}{2}\right)^2\right)\right] \hat{V}(x-y),$$

where $W_t(x) = (e^{itx} - 1)/ix$, and

$$K_{n,t}(x,y) = \int_{\mathbb{R}} \int_0^t \left[e^{is((x+v_1/2)^2-(z+v_1/2)^2)} + e^{is((x+v_2/2)^2-(z+v_2/2)^2)} \right] \cdot \hat{V}(x-z) K_{n-1,s}(z,y)\, ds\, dz. \tag{1}$$

$U_f(t, 0)$ will then be obtained as a norm-convergent series of the $K_{n,t}$'s, for small $|t|$, and extended to $U(t, s)$ using the propagator identity (cf. [4, Lemma 3.17]). Accordingly we will show in Theorem 1 that the operators $K_{n,t}$ are bounded in $L^2(\mathbb{R})$, and subsequently that

$$\|K_{n,t}\| \leq M[c(t)]^n,$$

where M is a constant independent of t and n, and $c(t) \leq K|t|$ (cf. Proposition 6). The rest of the method in [4, 3.8–3.12] will then complete the proof.

2. Existence of a Unitary Propagator

To simplify our exposition, we introduce the following notation; all of these expressions will arise in the proof of Theorem 1. When the limits of integration are omitted, the integration is understood to be over the real line:

$$\phi_{t,v}(x,y) = W_t\left(\left(x+\frac{v}{2}\right)^2 - \left(y+\frac{v}{2}\right)^2\right),$$

$$\tilde{\phi}_{t,v_1,v_2}(x,y,z) = W_t\left(\left(x+\frac{v_1}{2}\right)^2 - \left(y+\frac{v_2}{2}\right)^2 - (v_1-v_2)z - (v_1^2/4 - v_2^2/4)\right),$$

$$\tilde{K}_n(x,y) = \int \left| \int_0^t [e^{is((x+v_1/2)^2-(z+v_1/2)^2)} + e^{is((x+v_2/2)^2-(z+v_2/2)^2)}]\, K_{n-1,s}(z,y)\, ds \right| dz,$$

and

$$\bar{\bar{K}}^{1}_{n,t}(x, y, z) = \int \left| \int_0^t \left[e^{is((x+v_1/2)^2-(w+v_1/2)^2)} \right.\right.$$

$$\left.\left. + e^{is((x+v_1/2)^2-(w+v_2/2)^2-(v_1-v_2)z-(v_1^2/4-v_2^2/4))} \right] K_{n-1,s}(w, y)\, ds \right| dw;$$

$\bar{\bar{K}}^{2}_{n,t}(x, y, z)$ reverses the roles of v_1 and v_2 in $\bar{\bar{K}}^{1}_{n,t}(x, y, z)$. Note that

$$\bar{K}_{1,t}(x, y) = |\phi_{t,v_1}(x, y) + \phi_{t,v_2}(x, y)| \qquad \text{and}$$

$$\bar{\bar{K}}^{1}_{1,t}(x, y, z) = |\phi_{t,v_1}(x, y) + \tilde{\phi}_{t,v_1,v_2}(x, y, z)|.$$

We first prove a technical result.

LEMMA 1. $\sup_{x\in\mathbb{R}} \int |W_t(x - y^2)|\, dy < \infty$.

Proof. We already know that the result holds when the argument $x - y^2$ is replaced by $x^2 - y^2$; this is [4, Theorem 2.3]. In particular, $\sup_{x\geq 0} \int |W_t(x - y^2)|\, dy < \infty$. Consideration of the supremum over $x < 0$ is equivalent to

$$\sup_{x\in\mathbb{R}} \int \left| \frac{e^{-it(x^2+y^2)} - 1}{x^2 + y^2} \right| dy.$$

In this case, we simply note that the integrand is bounded by a constant multiple of $1/(y^2 + 1)$, and the proof is complete.

Our first main result is analogous to [4, Theorem 3.4], and establishes boundedness of the terms in the Dyson expansion for the interaction Hamiltonian $H_I(t)$.

THEOREM 1. *For $n \geq 1$, $t \in \mathbb{R}$,*

$$\text{(i)} \sup_{x\in\mathbb{R}} \int |K_{n,t}(x, y)|\, dy < \infty \qquad \textit{and} \qquad \text{(ii)} \sup_{x\in\mathbb{R}} \int |K^*_{n,t}(x, y)|\, dy < \infty.$$

In particular, for $n \geq 2$, $\sup_{x\in\mathbb{R}} \int |K_{n,t}(x, y)|\, dy \leq C(t)$

$$\cdot \max_{i=1,2} \left\{ \sup_{\substack{x\in\mathbb{R} \\ s\in[0,t]}} \int |\bar{K}_{n-1,s}(x, y)|\, dy,\ \sup_{x,z\in\mathbb{R}} \int |\bar{\bar{K}}^{i}_{n-1,t}(x, y, z)|\, dy \right\},$$

where $C(t)$ is a constant independent of n.

Proof. For $n = 1$, a change of variables shows that

$$\sup_{x\in R} \int |K_{1,t}(x, y)|\, dy \le 2 \sup_{x\in R} \int |W_t(x^2 - y^2)\hat{V}(x - y)|\, dy,$$

and the latter is finite by [4, Theorem 2.3].

Although it is not necessary, we consider the case $n = 2$ in some detail.

For the second-order term, we have after integrating by parts with respect to ds

$$\begin{aligned} K_{2,t}(x, y) &= \int \int_0^t e^{is(x^2-z^2)} [e^{iv_1 s(x-z)} + e^{iv_2 s(x-z)}]\, \hat{V}(x - z) K_{1,s}(z, y)\, ds\, dz \\ &= \int [\phi_{t,v_1}(x, z) + \phi_{t,v_2}(x, z)]\, \hat{V}(x - z) K_{1,t}(z, y)\, dz \\ &\quad - \int \int_0^t [\phi_{s,v_1}(x, z) + \phi_{s,v_2}(x, z)]\, \hat{V}(x - z) \\ &\quad [e^{is((z+v_1/2)^2-(y+v_1/2)^2)} + e^{is((z+v_2/2)^2-(y+v_2/2)^2)}] \cdot \hat{V}(z - y)\, ds\, dz. \end{aligned} \tag{2}$$

Noting that $K^*_{1,t}(x, y) = \overline{K_{1,t}(y, x)} = K_{1,t}(x, y)$ we see that the integral of the first term in (2), with respect to dy, is bounded by $[\sup_{x\in R} \int |K_{1,t}(x, y)|\, dy]^2$. Integrating the second term in (2) with respect to ds, we obtain

$$\begin{aligned} &\int \frac{\hat{V}(x - z)\hat{V}(z - y)}{i((x + v_1/2)^2 - (z + v_1/2)^2)} \Bigg[\phi_{t,v_1}(x, y) - \phi_{t,v_1}(z, y) \\ &\quad + W_t\left(\left(x + \frac{v_1}{2}\right)^2 - \left(y + \frac{v_2}{2}\right)^2 - (v_1 - v_2)z - \left(\frac{v_1^2}{4} - \frac{v_2^2}{4}\right)\right) \\ &\quad - \phi_{t,v_2}(z, y) \Bigg] dz \\ &+ \int \frac{\hat{V}(x - z)\hat{V}(z - y)}{i((x + v_2/2)^2 - (z + v_2/2)^2)} \Bigg[W_t\Bigg(\left(x + \frac{v_2}{2}\right)^2 \\ &\quad - \left(y + \frac{v_1}{2}\right)^2 - (v_2 - v_1)z \\ &\quad - \left(\frac{v_2^2}{4} - \frac{v_1^2}{4}\right)\Bigg) + \phi_{t,v_1}(z, y) - \phi_{t,v_2}(x, y) - \phi_{t,v_2}(z, y) \Bigg] dz, \end{aligned} \tag{3}$$

the complicated W_t-terms occurring as the "cross terms." It is clear that we need only consider one of the terms in (3) in detail.

Integrating the first term in (3) with respect to dy, we obtain

$$\int\int \frac{\hat{V}(x-z)\hat{V}(z-y)}{i((x+v_1/2)^2-(z+v_1/2)^2)}[\phi_{t,v_1}(x,y)+\tilde{\phi}_{t,v_1,v_2}(x,y,z) - (\phi_{t,v_1}(z,y)+\phi_{t,v_2}(z,y))]\,dz\,dy;$$

we claim that

$$\int \frac{|\hat{V}(x-z)\hat{V}(z-y)|}{|(x+v_1/2)^2-(z+v_1/2)^2|}|\phi_{t,v_1}(z,y)+\phi_{t,v_2}(z,y))|\,dy$$
$$\leq M^2\left[\sup_{x\in\mathbb{R}}\int|\tilde{K}_{1,t}(x,y)|\,dy + C_1(t)\right] \tag{4}$$
$$\frac{\max_{i=1,2}\left\{\sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}}\int|\tilde{K}_{1,s}(x,y)|\,dy,\ \sup_{x,z\in\mathbb{R}}\int|\tilde{\tilde{K}}{}^{i}_{1,t}(x,y,z)|\,dy\right\}}{|(x+v_1/2)^2-(z+v_1/2)^2|+1},$$

where the constant $C_1(t)$ is independent of x and z, and $|\hat{V}(x)| \leq M$, for all $x \in \mathbb{R}$. Now, if $|(x+v_1/2)^2-(z+v_1/2)^2| > 1$, then we use the fact that the integral in (4) is bounded by

$$\int|\tilde{\tilde{K}}{}^{i}_{1,t}(x,y,z)|dy + \int|\tilde{K}_{1,t}(z,y)|\,dy$$
$$\leq 2\max_{i=1,2}\left\{\int|\tilde{\tilde{K}}{}^{i}_{1,t}(x,y,z)|dy,\ \int|\tilde{K}_{1,t}(z,y)|\,dy\right\}.$$

We note that

$$\sup_{x\in\mathbb{R}}\int|\tilde{K}_{1,t}(x,y)|\,dy \leq 2\sup_{x\in\mathbb{R}}\int|W_t(x^2-y^2)|\,dy < \infty,$$

while

$$\sup_{x,z\in\mathbb{R}}\int|\tilde{\tilde{K}}{}^{i}_{1,t}(x,y,z)|\,dy$$
$$\leq \sup_{x\in\mathbb{R}}\int|\phi_{t,v_i}(x,y)|\,dy + \sup_{x,z\in\mathbb{R}}\int|\tilde{\phi}_{t,v_1,v_2}(x,y,z)|\,dy$$
$$= \sup_{x\in\mathbb{R}}\int|W_t(x^2-y^2)|\,dy + \sup_{x\in\mathbb{R}}\int|W_t(x-y^2)|\,dy < \infty.$$

Indeed, $(x + v_1/2)^2 + (v_1 - v_2)z - (v_1^2/4 - v_2^2/4)$ ranges over $\mathbb{R}$ when x and z do. Moreover,

$$\sup_{x\in\mathbb{R}} \int |W_t(x^2 - y^2)|\, dy \leq \sup_{x\in\mathbb{R}} \int |W_t(x - y^2)|\, dy,$$

and the latter is finite by Lemma 1.

Consequently, when $|(x + v_1/2)^2 - (z + v_1/2)^2| > 1$, the left-hand side of (4) is bounded by

$$CM^2 \frac{\max_{i=1,2}\left\{\int |\tilde{K}_{1,t}(z, y)|\, dy, \int |\tilde{\tilde{K}}{}^i_{1,t}(x, y, z)|\, dy\right\}}{|(x + v_1/2)^2 - (z + v_1/2)^2| + 1},$$

where C is independent of x and z.

In the event $|(x + v_1/2)^2 - (z + v_1/2)^2| \leq 1$, we undo the ds-integration and observe from (2) that the left-hand side of (4) is bounded by

$$\begin{aligned}
&\int |\phi_{t,v_1}(x, z)\hat{V}(x - z)K_{1,t}(z, y)|\, dy \\
&\quad + \int \left| \int_0^t [e^{is((x+v_1/2)^2-(z+v_1/2)^2)}]\, \hat{V}(x - z)K_{1,s}(z, y)\, ds \right| dy \\
&\leq 2tM^2 \sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}} \int |\tilde{K}_{1,s}(x, y)|\, dy,
\end{aligned}$$

so that in this case (4) is bounded by

$$\frac{4tM^2 \sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}} \int |\tilde{K}_{1,s}(x, y)|\, dy}{|(x + v_1/2)^2 - (z + v_1/2)^2| + 1}.$$

Consequently, we obtain from (2) (with $C(t) = 4t + C$ and $C_1(t) = 2C(t) \sup_{x\in\mathbb{R}} \int 1/(|x^2 - z^2| + 1)\, dz$; the factor of 2 occurs because of the latter two terms in (3))

$$\int |K_{2,t}(x, y)|\, dy \leq \left[\sup_{x\in\mathbb{R}} \int |K_{1,t}(x, y)|\, dy\right]^2$$
$$+ C_1(t)M^2 \max_{i=1,2} \left\{ \sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}} \int |\tilde{K}_{1,s}(x, y)|\, dy,\ \sup_{x,z\in\mathbb{R}} \int |\tilde{\tilde{K}}{}^i_{1,t}(x, y, z)|\, dy\right\} \tag{5}$$
$$\leq M^2 \left[\sup_{x\in\mathbb{R}} \int |\tilde{K}_{1,t}(x, y)|\, dy + C_1(t)\right]$$
$$\max_{i=1,2} \left\{ \sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}} \int |\tilde{K}_{1,s}(x, y)|\, dy,\ \sup_{x,z\in\mathbb{R}} \int |\tilde{\tilde{K}}{}^i_{1,t}(x, y, z)|\, dy\right\};$$

the interchange of the order of integration in (5) is justified because the iterated integral converges absolutely.

Note. Before turning to the remainder of the proof, it is convenient to observe at this point that in the case of N moving potentials, we obtain in place of (5)

$$\sup_{x\in\mathbb{R}} \int |K_{2,t}(x, y)|\, dy \leq M^2 \left[\sup_{x\in\mathbb{R}} \int |\tilde{K}_{1,t}(x, y)| + C_1(t)\right]$$
$$\cdot \max_{1\leq i\leq N} \left\{ \sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}} \int |\tilde{K}_{1,s}(x, y)|\, dy,\ \sup_{x,z\in\mathbb{R}} \int |\tilde{\tilde{K}}{}^i_{1,t}(x, y, z)|\, dy\right\},$$

where $\tilde{K}_{1,t}(x, y) = \sum_{j=1}^{N} \phi_{t,v_j}(x, y)$, $\tilde{\tilde{K}}{}^i_{1,s}(x, y, z) = \phi_{s,v_i} + \sum_{\substack{i=1\\ i\neq j}}^{N} \tilde{\tilde{\phi}}_{s,v_i,v_j}(x, y, z)$, and $C_1(t) = N(2Nt + C_1)$, where C_1 is independent of x, z, and t. Indeed, in place of (3) we have N integrals, each bounded by

$$\int \frac{|\hat{V}(x - z)\hat{V}(z - y)|}{|((x + v_i/2)^2 - (z + v_i/2)^2)|} [\tilde{\tilde{K}}{}^i_{1,t}(x, y, z) + \tilde{K}_{1,t}(z, y)]\, dz,$$

for $i = 1, \ldots, N$; these in turn are bounded in a manner analogous to (4).

Now, it remains to prove that for all $n \geq 2$,

$$\sup_{x\in\mathbb{R}} \int |K_{n,t}(x, y)|\, dy \leq M^2 \left[\sup_{x\in\mathbb{R}} \int |\tilde{K}_{1,t}(x, y)|\, dy + C_1(t)\right] \tag{6}$$
$$\cdot \max_{i=1,2} \left\{ \sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}} \int |\tilde{K}_{n-1,s}(x, y)|\, dy,\ \sup_{x,z\in\mathbb{R}} \int |\tilde{\tilde{K}}{}^i_{n-1,t}(x, y, z)|\, dy\right\}.$$

We have

$$
\begin{aligned}
K_{n,t}(x,y) &= \int\int_0^t [e^{is((x+v_1/2)^2-(z+v_1/2)^2)} + e^{is((x+v_2/2)^2-(z+v_2/2)^2)}] \\
&\quad \hat{V}(x-z)K_{n-1,s}(z,y)\,ds\,dz \\
&= \int [\phi_{t,v_1}(x,z) + \phi_{t,v_2}(x,z)]\hat{V}(x-z)K_{n-1,t}(z,y)\,dz \\
&\quad - \int\int_0^t [\phi_{s,v_1}(x,z) + \phi_{s,v_2}(x,z)]\hat{V}(x-z) \\
&\quad \int e^{is((z+v_1/2)^2-(w+v_1/2)^2)} + e^{is((z+v_2/2)^2-(w+v_2/2)^2)} \\
&\quad \hat{V}(z-w)K_{n-2,s}(w,y)\,ds\,dw\,dz \\
&= \int K_{1,t}(x,z)K_{n-1,t}(z,y)\,dz - \int \frac{\hat{V}(x-z)}{i((x+v_1/2)^2-(z+v_1/2)^2)} \\
&\quad \int\int_0^t [e^{is((x+v_1/2)^2-(w+v_1/2)^2)} + e^{is((x+v_1/2)^2-(z+v_1/2)^2+(z+v_2/2)^2-(w+v_2/2)^2)} \\
&\quad - e^{is((z+v_1/2)^2-(w+v_1/2)^2)} - e^{is((z+v_2/2)^2-(w+v_2/2)^2)}] \\
&\quad \hat{V}(z-w)K_{n-2,s}(w,y)\,ds\,dw\,dz \text{ minus a similar} \\
&\quad \text{term with } v_1 \text{ and } v_2 \text{ reversed.}
\end{aligned}
$$

As in the case $n = 2$, we first show that,

$$
\begin{aligned}
&\int\Bigg| \frac{\hat{V}(x-z)}{i((x+v_1/2)^2-(z+v_1/2)^2)} \int\int_0^t [e^{is((x+v_1/2)^2-(w+v_1/2)^2)} \\
&\quad + e^{is((x+v_1/2)^2-(z+v_1/2)^2+(z+v_2/2)^2-(w+v_2/2)^2)} \\
&\quad - e^{is((z+v_1/2)^2-(w+v_1/2)^2)} - e^{is((z+v_2/2)^2-(w+v_2/2)^2)}] \\
&\quad \hat{V}(z-w)K_{n-2,s}(w,y)\,ds\,dw \Bigg|\,dy \\
&\le \frac{M^2}{|(x+v_1/2)^2-(z+v_1/2)^2|} \int [|\tilde{\tilde{K}}{}^1_{n-1,t}(x,y,z) \\
&\quad - \tilde{K}_{n-1,t}(z,y)|]\,dy \\
&\le M^2 \left[\sup_{x\in\mathbb{R}} \int |\tilde{K}_{1,t}(x,y)|\,dy + C(t) \right] \\
&\quad \frac{\max_{i=1,2} \left\{ \sup_{\substack{x\in\mathbb{R} \\ s\in[0,t]}} \int |\tilde{K}_{n-1,s}(x,y)|\,dy,\ \sup_{x,z\in\mathbb{R}} \int |\tilde{\tilde{K}}{}^i_{n-1,t}(x,y,z)|\,dy \right\}}{|(x+v_1/2)^2-(z+v_1/2)^2|+1}.
\end{aligned}
\tag{7}
$$

Indeed, if $|(x + v_1/2)^2 - (z + v_1/2)^2| > 1$, the integral in (7) is bounded by

$$\int |\tilde{\tilde{K}}{}^{1}_{n-1,t}(x, y, z)|\, dy + \int |\tilde{K}_{n-1,t}(z, y)|\, dy$$

$$\leq 2 \max_{i=1,2} \left\{ \int |\tilde{\tilde{K}}{}^{i}_{n-1,t}(x, y, z)|\, dy, \int |\tilde{K}_{n-1,t}(z, y)|\, dy \right\}.$$

Consequently, in this case (7) is bounded by

$$\frac{C_1 M^2 \max_{i=1,2} \left\{ \int |\tilde{K}_{n-1,t}(z, y)|\, dy, \int |\tilde{\tilde{K}}{}^{i}_{n-1,t}(x, y, z|\, dt \right\}}{|(x + v_1/2)^2 - (z + v_1/2)^2| + 1},$$

where C_1 is independent of x and z. When $|(x + v_1/2)^2 - (z + v_1/2)^2| \leq 1$, we note that the left-hand side of (7) is bounded by

$$\int |\phi_{t,v}(x, z) \hat{V}(x - z) K_{n-1,t}(z, y)|\, dy$$

$$+ \int \left| \int_0^t [e^{is((x+v_1/2)^2-(z+v_1/2)^2)}]\, \hat{V}(x - z) K_{n-1,s}(z, y)\, ds \right| dy$$

$$\leq 2tM^2 \sup_{\substack{z\in\mathbf{R} \\ s\in[0,t]}} \int |\tilde{K}_{n-1,s}(z, y)|\, dy,$$

so that in this case (7) is bounded by

$$\frac{4tM^2 \sup_{\substack{z\in\mathbf{R} \\ s\in[0,t]}} \int |\tilde{K}_{n-1,s}(z, y)|\, dy}{|(x + v_1/2)^2 - (z + v_1/2)^2| + 1}.$$

Consequently, we obtain from the definition of $\tilde{K}_n$ that

$$\int |K_{n,t}(x, y)|\, dy \leq \left(\sup_{x\in\mathbf{R}} \int |K_{1,t}(x, y)|\, dy \right) \left(\sup_{x\in\mathbf{R}} \int |K_{n-1,t}(x, y)|\, dy \right)$$

$$+ 2C(t) M^2 \max_{i=1,2} \left\{ \sup_{\substack{x\in\mathbf{R} \\ s\in[0,t]}} \int |\tilde{K}_{n-1,s}(x, y)|\, dy, \sup_{x,z\in\mathbf{R}} \int |\tilde{\tilde{K}}{}^{i}_{n-1,t}(x, y, z)|\, dy \right.$$

$$\left.\int \frac{1}{|(x + v_i/2)^2 - (z + v_i/2)^2| + 1} dz\right\}$$

$$\leq M^2 \left[\sup_{x\in\mathbb{R}} \int |\tilde{K}_{1,t}(x, y)|\, dy + C_1(t)\right]$$

$$\cdot \max_{i=1,2} \left\{ \sup_{\substack{x\in\mathbb{R} \\ s\in[0,t]}} \int |\tilde{K}_{n-1,s}(x, y)|\, dy,\ \sup_{x,z\in\mathbb{R}} \int |\tilde{\tilde{K}}{}^{i}_{n-1,t}(x, y, z)|\, dy \right\},$$

where $C_1(t) = 2C(t) \sup_{x\in\mathbb{R}} \int (1/(|x^2 - z^2| + 1))\, dz$, and the proof of (i) is complete.

We shall prove (ii) by induction on n. Indeed, since $K_1(x, y) = K_1^*(x, y)$, it holds for $n = 1$. Assuming (ii) holds for $k \leq n - 1$, we then prove, exactly as in [4, Theorem 3.4 (iv)], the relation

$$K_{n,t}(x, y) = \sum_{j=1}^{n} (-1)^{j-1} K_{j,t}^* K_{n-j,t}(x, y). \tag{8}$$

Consequently (ii) holds for $k = n$, because by virtue of (8), it is equivalent to (i).

In order to prove that the time-ordered exponential series converges in the uniform operator topology to a unitary propagator for the time-dependent Hamiltonian, we next show the following:

PROPOSITION 2. *For all $n \geq 2$, $t \in \mathbb{R}$,*

$$\max_{i=1,2} \left\{\sup_{x\in\mathbb{R}} \int |\tilde{K}_{n,t}(x, y)|\, dy,\ \sup_{x,z\in\mathbb{R}} \int |\tilde{\tilde{K}}{}^{i}_{n,t}(x, y, z)|\, dy\right\}$$

$$\leq C_t \max_{i=1,2} \left\{ \sup_{\substack{x\in\mathbb{R} \\ s\in[0,t]}} \int |\tilde{K}_{n-1,s}(x, y)|\, dy, \right.$$

$$\sup_{x,z\in\mathbb{R}} \int |\tilde{\tilde{K}}{}^{i}_{n-1,t}(x, y, z)|\, dy,$$

$$\left. \sup_{x,w,z\in\mathbb{R}} \int |\tilde{\tilde{K}}{}^{i}_{n-1,t}(x, y, z, w)|\, dy \right\},$$

where $\tilde{\tilde{K}}{}^{1}_{n-1,t}(x, y, z, w)$ has the same kernel as $\tilde{\tilde{K}}_{n-1,t}(x, y, w)$, with a factor of $e^{is(w-z)(v_1-v_2)}$ in the integrand, and

$$C_t = M\left(\sup_{x\in\mathbb{R}} \int |\tilde{K}_{1,t}(x,y)|\,dy + C_2(t)\right),$$

$$\text{with } C_2(t) = 2(4t + C) \sup_{x\in\mathbb{R}} \int \frac{1}{|x - z^2| + 1}\,dz.$$

Proof. It is clear that the inequality holds with $\sup_{x\in\mathbb{R}} \int |\tilde{K}_{n,t}(x, y|\,dy$ on the left-hand side, for we simply remove the constants M that bound $\hat{V}$. In order to complete the proof, we must show that in fact (6) holds with $\bar{\tilde{K}}^i_{n,t}(x, y, z)$ on the left-hand side. Indeed,

$$\begin{aligned}\bar{\tilde{K}}^1_{n,t}(x,y,z) &= \int \left| \int_0^t [e^{is((x+v_1/2)^2-(w+v_1/2)^2)} \right.\\ &\quad \left. + e^{is((x+v_1/2)^2-(w+v_2/2)^2)-(v_1-v_2)z-(v_1^2/4-v_2^2/4))}]\, K_{n-1,s}(w,y)\,ds \right| dw\\ &\le \int \bar{\tilde{K}}^1_{1,t}(x,w,z)|K_{n-1,t}(w,y)|\,dw\\ &\quad + \int \frac{M}{|(x+v_1/2)^2-(w+v_1/2)^2|}[\bar{\tilde{K}}^1_{n-1,t}(x,y,w) + \tilde{K}_{n-1,t}(w,y)]\,dw\\ &\quad + \int \frac{M}{|(x+v_1/2)^2-(w+v_2/2)^2-(v_1-v_2)z-(v_1^2/4-v_2^2/4)|}\\ &\quad\quad [\bar{\tilde{K}}^1_{n-1,t}(x,y,z,w) + \tilde{K}_{n-1,t}(w,y)]\,dw.\end{aligned}$$

Each of these integrals is bounded precisely as in (7), except that the constant $C_2(t)$ involves the quantity

$$\sup_{x,z\in\mathbb{R}} \int \frac{1}{|(x+v_1/2)^2-(w+v_2/2)^2-(v_1-v_2)z-(v_1^2/4-v_2^2/4)|+1}\,dw,$$

which is finite for reasons discussed in the proof of Lemma 1.

Remark. We see from the proof of Theorem 1 that analogous results are valid if we integrate from s to t, with the exception that the bound $C(t)$ is $4|t - s| + C_1$. The corresponding integral operators will be denoted $K_{n,t,s}$.

COROLLARY 3. *For all $n \ge 2$, $t \in \mathbb{R}$,*

$$\sup_{x\in\mathbb{R}} \int |\tilde{K}_{n,t}(x,y)|\,dy \le c^{n-1}\cdot K,$$

where c and K are constants independent of n, and are bounded for $|t|$ bounded.

Proof. By repeated application of Theorem 1 and Proposition 2, we obtain

$$\int |\tilde{K}_{n,t}(x,y)|\,dy \leq (c_t)^{n-1} \cdot \max_{i=1,2} \Bigg\{ \sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}} \int |\tilde{K}_{1,s}(x,y)|\,dy,$$

$$\sup_{\substack{x,z\in\mathbb{R}\\ s\in[0,t]}} \int |\tilde{\tilde{K}}{}^{i}_{1,s}(x,y,z|\,dy,\ \sup_{\substack{x,z,w\in\mathbb{R}\\ s\in[0,t]}} \int |\tilde{\tilde{K}}{}^{i}_{1,s}(x,y,z,w)|\,dy, \Bigg\},$$

where $c_t = 2(4|t| + C)\sup_{x\in\mathbb{R}} \int (1/(|x - z^2| + 1))\,dz + \sup_{x\in\mathbb{R}} \int |\tilde{K}_{1,t}(x, y)|\,dy$. Indeed,

$$\tilde{\tilde{K}}{}^{1}_{n}(x,y,z,w) \leq \int \tilde{\tilde{K}}{}^{1}_{1,t}(x,u,z,w)\tilde{K}_{n-1,t}(u,y)\,du$$

$$+ \int \frac{M}{|(x + v_1/2)^2 - (u + v_1/2)^2 + (w - z)(v_1 - v_2)|}$$

$$\left[\dot{\tilde{\tilde{K}}}{}^{1}_{n-1,t}(x,y,u) + \dot{\tilde{K}}_{n-1,t}(u,y)\right] du$$

$$+ \int \frac{M}{|(x + v_1/2)^2 - (u + v_2/2)^2 + z(v_1 - v_2) - (v_1^2/4 - v_2^2/4)|}$$

$$\left[\tilde{\tilde{K}}{}^{1}_{n-1,t}(x,y,z,u) + \dot{\tilde{K}}_{n-1,t}(u,y)\right] du,$$

where the $\cdot$ denotes a multiple of $e^{is(w-z)}$ in the integrand of the kernel, and the latter does not affect the bounds.

In order to show that the time-ordered series $\sum_{n=1}^{\infty} (i)^n K_{n,t,s}$ converges, we must specify how the bound in Corollary 3 depends on $|t - s|$; this is somewhat more delicate than in [3]. To this end, we introduce a new family of auxiliary operators. For s, $t \in \mathbb{R}$, let

$$L^s_{1,t}(x,y) = [e^{is((x+v_1/2)^2-(y+v_1/2)^2)}\phi_{t,v_1}(x,y)$$

$$+ e^{is((x+v_2/2)^2-(y+v_2/2)^2)}\phi_{t,v_2}(x,y)]\,\hat{V}(x-y),$$

and for $n \geq 2$,

$$L^s_{n,t}(x,y) = \int\int_0^t [e^{i(u+s)((x+v_1/2)^2-(z+v_1/2)^2)} + e^{i(u+s)((x+v_2/2)^2-(z+v_2/2)^2)}]$$

$$\hat{V}(x-z)L^s_{n-1,u}(z,y)\,dz\,du.$$

We then have

LEMMA 4. *For* $s, t \in \mathbb{R}$, $n \geq 1$,

$$K_{n,t,s}(x, y) = L^s_{n,t-s}(x, y).$$

Proof. For $n = 1$, the result follows immediately.

$$\begin{aligned} K_{1,t,s}(x, y) &= \left[\frac{e^{it((x+v_1/2)^2-(y+v_1/2)^2)} - e^{is((x+v_1/2)^2-(y+v_1/2)^2)}}{i((x+v_1/2)^2-(y+v_1/2)^2)} \right. \\ &\quad \left. + \frac{e^{it((x+v_2/2)^2-(y+v_2/2)^2)} - e^{is((x+v_2/2)^2-(y+v_2/2)^2)}}{i((x+v_2/2)^2-(y+v_2/2)^2)} \right] \hat{V}(x-y) \\ &= L^s_{1,t-s}(x, y), \end{aligned}$$

and the inductive step merely requires a change of variables. We omit the details.

Next, we observe that the entire development (in particular, Theorem 1), is valid for $L^s_{n,t}$. Indeed, the crucial bounds in (4) are still valid, for the occurrence of the factors $e^{is((x+v_i/2)^2-(y+v_i/2)^2)}$, $i = 1, 2$, is immaterial when we take absolute values.

We now introduce operators analogous to $K'_{n,t}$ in [4, Definition 3.8]: for $s, u \in \mathbb{R}$, $t > 0$.

$$\begin{aligned} L'^s_{1,t}(u, x, y) &= [e^{is((x/\sqrt{t}+v_1/2)^2-(y/\sqrt{t}+v_1/2)^2)}\phi_{u,\sqrt{tv_1}}(x, y) \\ &\quad + e^{is((x/\sqrt{t}+v_2/2)^2-(y/\sqrt{t}+v_2/2)^2)}\phi_{u,\sqrt{tv_2}}(x, y)]\, \hat{V}\left(\frac{x-y}{\sqrt{t}}\right), \end{aligned}$$

and for $n \geq 2$,

$$\begin{aligned} L'^s_{n,t}(u, x, y) &= \int_0^u \int [e^{i(s+wt)((x/\sqrt{t}+v_1/2)^2-(z/\sqrt{t}+v_1/2)^2)} \\ &\quad + e^{i(s+wt)((x/\sqrt{t}+v_2/2)^2-(z/\sqrt{t}+v_2/2)^2)}]\, \hat{V}\left(\frac{x-y}{\sqrt{t}}\right) \cdot L'^s_{n-1,t}(w, z, y)\, dz\, dw. \end{aligned}$$

We then have

LEMMA 5. *For all* $n \geq 1$, $r, s, t, x, y \in \mathbb{R}$,

$$L^s_{n,tr}(x, y) = (t^{1/2})^{n+1} L'^s_{n,t}(r, t^{1/2}x, t^{1/2}y).$$

The proof is a straightforward induction on n.

Remarks. (1) Lemmas 4 and 5 together yield the relationship

$$K_{n,t,s}(x,y) = [(t-s)^{1/2}]^{n+1} L'^{s}_{n,t-s}(1, (t-s)^{1/2}x, (t-s)^{1/2}y), \qquad t-s>0,$$

which will be useful in establishing convergence of the Dyson expansion.

(2) Theorem 1 remains valid for the operators $L'^{s}_{n,t}$, as observed following Lemma 4.

Consequently, we have

PROPOSITION 6. *For all $n \geq 1$, t, $s \in \mathbb{R}$,*

$$\sup_{x\in\mathbb{R}} \int |K_{n,t,s}(x,y)|\, dy \leq M(\tilde{c})^{n-1}(|t-s|^{1/2})^n,$$

where M and $\tilde{c}$ are constants independent of n, t, and s, and similarly for $K^*_{n,t,s}$.

Proof. If $t-s \geq 0$,

$$\begin{aligned}\int |K_{n,t,s}(x,y)|\, dy &= \int |L^{s}_{n,t-s}(x,y)|\, dy \\ &= [(t-s)^{1/2}]^{n+1} \int |L'^{s}_{n,t}(1, (t-s)^{1/2}x, (t-s)^{1/2}y)|\, dy \\ &= [(t-s)^{1/2}]^{n} \int |L'^{s}_{n,t}(1, (t-s)^{1/2}x, y)|\, dy,\end{aligned}$$

so that

$$\sup_{x\in\mathbb{R}} \int |K_{n,t,s}(x,y)|\, dy \leq M(\tilde{c})^{n-1}(|t-s|^{1/2})^n,$$

where $\tilde{c} = K + \sup_{x\in\mathbb{R}} \int |L'^{s}_{1,t}(1,x,y)|\, dy$, and

$$M = \max_{i=1,2} \left\{ \sup_{\substack{x\in\mathbb{R} \\ u\in[0,t]}} \int |\tilde{L}'^{s}_{n,t}(u,x,y)|\, dy,\ \sup_{\substack{x,z\in\mathbb{R} \\ u\in[0,t]}} \int |\tilde{\tilde{L}}'^{s}_{1,t}(x,y,z)\, dy \right\}.$$

From [4, Proposition 3.11], we obtain the general result.

We now follow the development in [4, Sects. 3.14–3.19], and define $U_I(t,s)$ as a norm-convergent time-ordered exponential series, first for small $|t-s|$, and then via the propagator identity for all $t, s \in \mathbb{R}$. We have

THEOREM 7. *For $t, s \in \mathbb{R}$, with $|t - s|$ sufficiently small, the series*

$$\sum_{n=1}^{\infty} (i)^n K_{n,t,s}$$

converges in the uniform operator topology to an integral operator $U(t, s)$ with kernel

$$K(x, y) = \sum_{n=1}^{\infty} (i)^n K_{n,t,s}(x, y).$$

Using the propagator identity $U(t, r)U(r, s) = U(t, s)$, we are able to extend the definition of U_I to arbitrary values of s and t (cf. [4, Lemma 3.17]). Consequently, we obtain

THEOREM 8. *For all $r, s, t \in \mathbb{R}$, the following hold:*

(i) $U_I(t, s)^* = U_I(s, t)$,

(ii) $U_I(t, r)U_I(r, s) = U_I(t, s)$,

(iii) *the mapping $(t, s) \mapsto U_I(t, s)$ is jointly continuous from $\mathbb{R}^1 \times \mathbb{R}^1$ into the set of unitary operators on $L^2(\mathbb{R})$.*

COROLLARY 9. *The unitary propagator $U(t, 0)$ for the Hamiltonian $H_0 + \mathbb{V}(t)$ is given by $e^{itH_0}U_I(t, 0)$.*

3. SINGULAR PERTURBATIONS OF $(1/i)$ (d/dx)

Segal has discussed in [9] a method for defining a self-adjoint operator that corresponds formally to $(1/i)(d/dx) + c\delta$, $c \in \mathbb{R}$. If we let H = Heaviside function, then $e^{-icH}(1/i)(d/dx)e^{icH}$ is precisely $(1/i)(d/dx) + cH'$ when H is smooth, and depends continuously on H in the sense that when $H_n \to H$ in the sense of distributions, then the corresponding unitary groups (which are equal to $e^{-icH_n}U(t)e^{icH_n}$, where $U(t)$ is translation on the real line) converge in the strong operator topology. The domain of this self-adjoint operator is specified by the boundary condition $\phi(0^+) = e^{ic}\phi(0^-)$. It is an easy matter to define, in the spirit of section 2, *time-dependent* δ-potentials and their corresponding propagators. Indeed, for $v, t \in \mathbb{R}$, let $H(t) = e^{-icH(x-vt)}(1/i)(d/dx)e^{icH(x-vt)}$. For each fixed t, $H(t)$ is a self-adjoint operator on $L^2(\mathbb{R})$, with domain determined by the boundary condition $\phi(vt^+) = e^{ic}\phi(vt^-)$. Formally, this operator corresponds to $(1/i)(d/dx) + c\delta(x - vt)$ (cf. [1] for a discussion that provides a more plausible interpretation of $H(t)$ as $(1/i)(d/dx) + 2\tan(c/2)\delta(x - vt)$).

Using the interaction representation, we see that, at least in the event that H is a smooth real-valued measurable function with bounded derivative, the propagator for $H(t)$ is $U(t, 0) = U_0(t)U_I(t, 0)$, where $U_I(t, 0)$ is the propagator for the interaction Hamiltonian $H_I(t) = cU_0(-t)H'(x - vt)U_0(t)$. In this tractable case, $U_I(t, 0) = e^{i\int_0^t H_I(s)ds}$, and so for $\phi \in L^2(\mathbb{R})$, we see easily that

$$U_I(t, 0)\phi(x) = e^{(-ic/(1+v))[H(x-(v+1)t)-H(x)]}\phi(x),$$

so that

$$U(t, 0)\phi(x) = e^{(-ic/(1+v))[H(x-vt)-H(x+t)]}\phi(x + t).$$

This is a unitary propagator for the singular case where H = Heaviside function in the sense that if $\{H_n\}$ is a sequence of real-valued functions converging to the δ-function in the sense of distributions, then $U_n(t, 0) \to U(t, 0)$ in the strong operator topology. It is clear that the case of finitely many moving delta potentials can be handled in a similar fashion, with propagator

$$U(t, 0)\phi(x) = \exp\left[-i\sum_{i=1}^{n}\frac{c_i}{1+v_i}[H(x - v_it) - H(x + t)]\right]\phi(x + t).$$

We remark that the propagator may also be obtained by the method in [2], where it is observed that

$$U(t, 0) = T(t)e^{-it\tilde{H}},$$

with $T(t)f(x) = f(x - vt)$, and

$$\tilde{H} = \frac{(1 - v)}{i}\frac{d}{dx} + c\delta = e^{(-ic/(1-v))H}\frac{(1 - v)}{i}\frac{d}{dx}e^{(ic/(1-v))H}.$$

(*Note:* This in fact gives the propagator for $-H(t)$.)

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